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Pure spinor partition function using Padé approximants

E. Aldo Arroyo

Instituto de Física Teórica, São Paulo State University,
Rua Pamplona 145, São Paulo, SP 01405-900, Brasil
E-mail: aldohep@ift.unesp.br

Abstract: In a recent paper, the partition function (character) of ten-dimensional pure spinor worldsheet variables was calculated explicitly up to the fifth mass-level. In this letter, we propose a novel application of Padé approximants as a tool for computing the character of pure spinors. We get results up to the twelfth mass-level. This work is a first step towards an explicit construction of the complete pure spinor partition function.

Keywords: Superstrings and Heterotic Strings, Topological Strings.

Contents

1. Introduction

A few years ago, a new formalism was proposed for quantizing the superstring in a manifestly ten-dimensional super-Poincaré covariant manner [1]. This formalism has been used for computing covariant multiloop amplitudes [2], leading to new insights into perturbative finiteness of superstring theory [3].

This pure spinor formalism for superstrings, in the flat background, is based on a set of ten-dimensional superspace variables x^m, θ^α (and its conjugate p_α), and a set of bosonic ghosts fields λ^α (and its conjugate w_α) transforming as SO(10) spinors and satisfying a pure spinor constraint $\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0$ for $m = 1$ to 10. The need for this constraint comes from nilpotency of a BRST operator constructed from fermionic Green-Schwarz (GS) constraints and the pure spinor λ^α. It has been shown that the string spectrum found using the BRST cohomology coincides with the light-cone spectrum of the GS string [4–6]. This fact was also recently proven by using the pure spinor superstring partition function [7] (up to the fifth mass-level).

The information about the spectrum is encoded in the partition function

$$Z(q,t) \equiv Tr_{\mathcal{H}} \left[(-)^F q^{L_0} t^{J_0} \right] = \sum_h Z_h(t) q^h \,, \tag{1.1}$$

where $J_0 = \oint dz J(z)$ and $L_0 = \oint dz\, zT(z)$, J_0 is the zero mode of the U(1) current $J(z)$, L_0 is the zero mode of the Virasoro generator $T(z)$, F the fermion number (which takes account of the statistics), and $\mathcal{H}$ the corresponding Hilbert space. The computation of the pure spinor superstring partition function boils down to compute the partition function of the pure spinor sector since the calculation of the matter sector (x^m, θ^α) is very simple [7–9].

There have been many attempts for computing the partition function of pure spinors, for instance, in [8, 9] character formulas $Z_h(t)$ were computed up to the level $h = 1$, i.e. the zero mode $Z_0(t)$ and first massive $Z_1(t)$ part of the partition function (1.1). In [10, 11], the authors have tried to find a prescription which allows to compute the partition function of curved beta-gamma systems, but the employment of those techniques to the case of pure spinors are still lacking. In [12] it was studied partition functions of certain subset of curved beta-gamma systems (defined by quadratic constraints) as toy models in order to specialize then to the case of pure spinors.

Recently, it has been presented a formal expression for the partition function of pure spinors in terms of an infinite set of free-field ghosts [7]

$$Z(q,t) = \prod_{k=1}^{\infty}\left[(1-t^k)^{-N_k}\prod_{h=1}^{\infty}(1-q^h t^k)^{-N_k}(1-q^h t^{-k})^{-N_k}\right], \tag{1.2}$$

where N_k are the multiplicities of the ghost fields. The use of these ghosts comes from the resolution of the pure spinor constraint, and the necessity of infinitely many of them is because the pure spinor constraint is infinitely reducible [7, 13]. Although it may seem difficult to extract useful information from the formal expression (1.2), by appealing to some regularization procedure in order to guarantee the convergence of the infinite product over k, character formulas $Z_h(t)$ were calculated up to the fifth mass-level ($h = 5$). The same character formulas were also computed using fixed point techniques [7].

In this paper, we propose a novel application of Padé approximants [14] as a tool for computing higher-level character formulas $Z_h(t)$ of pure spinors. We show by explicit computation that our first five character formulas are in agreement with the ones found in [7], and in addition, we obtain results up to the twelfth mass-level ($h = 12$). It is left as an ambitious challenge to find the complete pure spinor partition function. We hope that our work is a first step towards this explicit construction of the pure spinor partition function.

The paper is organized as follows: In section 2, a short review of SO(10) pure spinors is given. In section 3, we review some concepts and ideas regarding to the partition function of pure spinors. In section 4, we describe the method of Padé approximants for computing higher level character formulas, and we present our results. A summary and further interesting topics are given in section 5. Finally in appendix A, we present some details involved in the computations of higher level character formulas.

2. Pure spinors: a reminder

The SO(10) pure spinor λ^α is constrained to satisfy $\lambda^\alpha\gamma^m_{\alpha\beta}\lambda^\beta = 0$, where $m = 1$ to 10, α, $\beta = 1$ to 16. $\gamma^m_{\alpha\beta}$ and $\gamma^{m\,\alpha\beta}$ are 16×16 symmetric matrices which are the off-diagonal blocks of the 32×32 ten-dimensional Γ-matrices and satisfy $\gamma^{(m}_{\alpha\gamma}\gamma^{n)\,\gamma\beta} = 2\eta^{mn}\delta^\beta_\alpha$. This implies that $\lambda^\alpha\lambda^\beta$ can be written as

$$\lambda^\alpha\lambda^\beta = \frac{1}{5!2^5}\gamma^{\alpha\beta}_{mnpqr}(\lambda^\gamma\gamma^{mnpqr}_{\gamma\delta}\lambda^\delta) \tag{2.1}$$

where $\lambda\gamma^{mnpqr}\lambda$ defines a 5-dimensional complex hyperplane. This 5-dimensional complex hyperplane is preserved up to a phase by a U(5) subgroup of SO(10) rotations. So projective pure spinors in $d = 10$ Euclidean dimensions parameterize the coset space SO(10)/ U(5), which implies that λ^α has 11 independent complex degrees of freedom [15].

The pure spinor constraint (2.1) can be solved and we can express λ^α in terms of the 11 independent degrees of freedom. This was done in [1] by decomposing the 16 components of λ^α into SU(5) × U(1) representations as

$$\lambda^+ = \gamma, \quad \lambda_{[ab]} = \gamma u_{[ab]}, \quad \lambda_{[abcd]} = -\frac{1}{8}\gamma u_{[ab}u_{cd]}, \tag{2.2}$$

where γ is an SU(5) scalar, and $u_{[ab]}$ is an SU(5) antisymmetric two-form. Using this decomposition, and by bosonizing the (β, γ) fields as $(\beta = \partial\xi e^{-\phi}, \gamma = \eta e^{\phi})$, we can write the formulas for the currents [8]

$$\begin{aligned} J &= -\frac{5}{2}\partial\phi - \frac{3}{2}\eta\xi, \\ N^{ab} &= v^{ab}, \\ N_a^b &= -u_{ac}v^{bc} + \delta_a^b\left(\frac{5}{4}\eta\xi + \frac{3}{4}\partial\phi\right), \\ N_{ab} &= 3\partial u_{ab} + u_{ac}u_{bd}v^{cd} + u_{ab}\left(\frac{5}{2}\eta\xi + \frac{3}{2}\partial\phi\right), \\ T &= \frac{1}{2}v^{ab}\partial u_{ab} - \frac{1}{2}\partial\phi\partial\phi - \eta\partial\xi + \frac{1}{2}\partial(\eta\xi) - 4\partial(\partial\phi + \eta\xi), \end{aligned} \tag{2.3}$$

where T is the stress-energy tensor and the worldsheet fields satisfy the OPE's

$$\eta(y)\xi(z) \sim (y-z)^{-1}, \quad \phi(y)\phi(z) \sim -\log(y-z), \quad v^{ab}u_{cd} \sim \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}. \tag{2.4}$$

Using these parameterizations of a pure spinor, the OPE's of the currents in (2.3) can be computed to be

$$\begin{aligned} N_{mn}(y)\lambda^\alpha(z) &\sim \frac{1}{2}\frac{1}{y-z}(\gamma_{mn}\lambda)^\alpha, \qquad J(y)\lambda^\alpha(z) \sim \frac{1}{y-z}\lambda^\alpha, \\ N^{kl}(y)N^{mn}(z) &\sim -\frac{3}{(y-z)^2}(\eta^{n[k}\eta^{l]m}) + \frac{1}{y-z}(\eta^{m[l}N^{k]n} - \eta^{n[l}N^{k]m}), \\ J(y)J(z) &\sim -\frac{4}{(y-z)^2}, \qquad J(y)N^{mn}(z) \sim 0, \\ N_{mn}(y)T(z) &\sim \frac{1}{(y-z)^2}N_{mn}(z), \qquad J(y)T(z) \sim -\frac{8}{(y-z)^3} + \frac{1}{(y-z)^2}J(z), \\ T(y)T(z) &\sim \frac{1}{2}\frac{22}{(y-z)^4} + \frac{2}{(y-z)^2}T(z) + \frac{1}{y-z}\partial T. \end{aligned} \tag{2.5}$$

So the conformal central charge is 22, the ghost-number anomaly is −8, the Lorentz central charge is −3, and the ghost-number central charge is −4. One can verify the consistency of these charges by considering the Sugawara construction of the stress tensor

$$T = \frac{1}{2(k+\delta)}N_{mn}N^{mn} + \frac{1}{8}JJ + \partial J\,, \tag{2.6}$$

where k is the Lorentz central charge, δ is the dual Coxeter number for SO(10). Setting $k = -3$, one finds that the SO(10) current algebra contributes −27 to the conformal central charge and the ghost current contributes 49 to the conformal central charge. So the total conformal central charge is 22 as expected.

3. Partition function of pure spinors

In this section, we review some aspects of the pure spinor partition function and indicate the main results we are going to use. The characters of the states (in the pure spinor Hilbert space [7]) we wish to keep track of are their statistics, weights h (Virasoro levels), t-charge (measured by $J = w_\alpha \lambda^\alpha$). Introducing formal variables (q, t) for each quantum number, we define the character as

$$\begin{aligned} Z(q,t) &= \mathrm{Tr}(-1)^F q^{L_0} t^{J_0} \\ &= \sum_{h\geq 0} Z_h(t) q^h \\ &= \sum_{h\geq 0,n} \mathcal{N}_{h,n} q^h t^n \,. \end{aligned} \tag{3.1}$$

The trace is taken over the pure spinor Hilbert space. The quantum numbers of the basic operators ω and λ are: $h(\omega, \lambda) = (1, 0)$, $t(\omega, \lambda) = (-1, 1)$. In [7], it was given a prescription for computing the $\mathcal{N}_{h,n}$ coefficients.[1] Once these coefficients are known, it is possible to calculate the character of pure spinors at each level h

$$Z_h(t) = \sum_n \mathcal{N}_{h,n} t^n \,. \tag{3.2}$$

At the lowest level, the Hilbert space is spanned by non-vanishing polynomials of λ. Due to the pure spinor constraint, λ's can only appear in the "pure spinor representations"

$$\lambda^{((\alpha_1}\lambda^{\alpha_2}\cdots\lambda^{\alpha_n))} = [0000n]t^n \,, \quad (n\geq 0)\,. \tag{3.3}$$

Here, we also indicated the t-charge of the state, and the symbol $((\alpha_1\alpha_2\cdots\alpha_n))$ signifies the "spinorial γ-traceless condition", which means that the expression is zero when any two indices $\alpha_i\alpha_j$ are contracted using $\gamma^\mu_{\alpha_i\alpha_j}$. Since the pure spinor representations have dimension

$$\dim[0000n] = \frac{(n+7)(n+6)(n+5)^2(n+4)^2(n+3)^2(n+2)(n+1)}{7\cdot 6\cdot 5^2\cdot 4^2\cdot 3^2\cdot 2}\,, \tag{3.4}$$

the level zero character is easily found to be [8, 9]

$$Z_0(t) = \frac{1-10t^2+16t^3-16t^5+10t^6-t^8}{(1-t)^{16}}\,. \tag{3.5}$$

As it was analyzed in [7, 13], the pure spinor constraint can be resolved by introducing an infinite chain of free-field ghosts. The multiplicities N_k of the ghosts can be obtained by writing the level zero character (3.5) as [7, 8]

$$Z_0(t) = \prod_{k=1}^{\infty}(1-t^k)^{-N_k}\,. \tag{3.6}$$

[1] These coefficients represent the number of states at each level h with n t-charge, and its sign tell us whether the states are fermionic or bosonic.

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The fields at ghost number k will have $|N_k|$ components, and will be bosons for $N_k > 0$ and fermions for $N_k < 0$. The multiplicities N_k contain the information about the Virasoro central charge, as well as the ghost current algebra:

$$\frac{1}{2}c_{\rm vir} = \sum_{k=1}^{\infty} N_k, \qquad a_{\rm ghost} = -\sum_{k=1}^{\infty} kN_k, \qquad c_{\rm ghost} = -\sum_{k=1}^{\infty} k^2 N_k. \tag{3.7}$$

We can easily deduce from (3.5) and (3.6):

$$N_1 = 16,\ N_2 = -10,\ N_3 = 16,\ N_4 = -45,\ N_5 = 144,\ N_6 = -456,\ N_7 = 1440,\ \ldots \tag{3.8}$$

For the computations to be done in the appendix A, we will need to know the value of the moments of the N_k's, i.e. we want: $\sum_{k=1}^{\infty} k^{s+1} N_k$. This was analyzed in [8]. The moments of (3.8) are given by

$$\begin{aligned}\sum_{k=1}^{\infty} k^{s+1} N_k &= 12 - 2^{s+1} - \frac{1}{\zeta(-s)} \sum_{k=1}^{\infty} k^s \left((-2-\sqrt{3})^k + (-2+\sqrt{3})^k\right) \\ &= 12 - 2^{s+1} - \frac{Li_{-s}(-2-\sqrt{3}) + Li_{-s}(-2+\sqrt{3})}{\zeta(-s)},\end{aligned} \tag{3.9}$$

where $Li_s(z)$ is the so-called polylogarithm (also known as de Jonquière's function), it is a special function defined by the sum

$$Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

Another way to get $\sum_{k=1}^{\infty} k^{s+1} N_k$ is by considering the general expression of the form we analyzed in (3.5) and (3.6):

$$\prod_{k=1}^{\infty} (1-t^k)^{-N_k} = \frac{P(t)}{Q(t)},$$

where P and Q are some polynomials. We have

$$\sum_{k=1}^{\infty} N_k \log(1-\mathrm{e}^{kx}) = -\log \frac{P(\mathrm{e}^x)}{Q(\mathrm{e}^x)}.$$

Since

$$\log(1-\mathrm{e}^x) = \log(-x) + \frac{x}{2} + \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g)!} x^{2g},$$

where B_k are Bernoulli numbers, we have:

$$\begin{aligned}\log(x) \sum_{k=1}^{\infty} N_k + \sum_{k=1}^{\infty} \log(-k) N_k + \frac{x}{2} \sum_{k=1}^{\infty} kN_k + \\ \sum_{g=1}^{\infty} \frac{B_{2g}}{2g(2g)!} x^{2g} \sum_{k=1}^{\infty} k^{2g} N_k = -\log \frac{P(\mathrm{e}^x)}{Q(\mathrm{e}^x)}.\end{aligned} \tag{3.10}$$

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Using (3.5) and expanding the right hand side r.h.s. of (3.10), we obtain the following value for the moments

$$\sum_{k=1}^{\infty} N_k = 11\ , \qquad \sum_{k=1}^{\infty} kN_k = 8\ , \qquad \sum_{k=1}^{\infty} k^2 N_k = 4\ , \qquad \sum_{k=1}^{\infty} k^4 N_k = -4\ , \tag{3.11}$$
$$\sum_{k=1}^{\infty} k^6 N_k = 4\ , \qquad \sum_{k=1}^{\infty} k^8 N_k = \frac{68}{3} \qquad \text{and} \qquad \sum_{k=1}^{\infty} k^{10} N_k = -396\,.$$

The first three moments of N_k's given in (3.11) contain the information about the conformal central charge c_{vir}, the ghost-number anomaly a_{ghost}, and the ghost-number central c_{ghost} (3.7). As we can check, the value of these current central charges are in agreement with the non-covariant calculation (2.5).

In a recent paper [7], using the ghosts, it was constructed a BRST operator and since this operator was given such that it carries zero t-charge, the partition function of its cohomology is equal to that of the total Hilbert space of (now unconstrained) pure spinors and the ghosts. Therefore, the partition function of pure spinors was formally written as

$$Z(q,t) = \prod_{k=1}^{\infty}\left[(1-t^k)^{-N_k}\prod_{h=1}^{\infty}(1-q^h t^k)^{-N_k}(1-q^h t^{-k})^{-N_k}\right] \tag{3.12}$$
$$Z(q,t) = \prod_{k=1}^{\infty}\left[\prod_{h=0}^{\infty}(1-q^h t^k)^{-N_k}\prod_{h=1}^{\infty}(1-q^h t^{-k})^{-N_k}\right] = \sum_{h=0}^{\infty} Z_h(t) q^h.$$

Using (3.11) and (3.12), an elementary calculation shows

$$Z(q,t) = -t^{-8} Z(q, 1/t), \tag{3.13}$$
$$Z(q,t) = -t^{-4} q^2 Z(q, q/t)\,. \tag{3.14}$$

These symmetries (3.13) and (3.14) of the partition function are referred as field-antifield and $*$-conjugation symmetry respectively [7]. In the forthcoming sections, starting from the formal expression of the partition function (3.12), we are going to describe a method for computing higher level character formulas $Z_h(t)$.

4. Padé approximants

The Padé approximation seeks to approximate the behavior of a function by a ratio of two polynomials. This ratio is referred to as the Padé approximant. This approximation works nicely even for functions containing poles, because the use of rational functions allows them to be well-represented. Recently, the Padé approximation has been applied to string field theory to analyze the tachyon condensation [16–18].

Let us now consider the general equations of the Padé approximation. Given some function $f(t)$, its $[M/N]$ Padé approximant denoted by $f^{[M/N]}(t)$ is a rational function of the form [14]

$$f^{[M/N]}(t) = \frac{1+\sum_{j=1}^{M} p_j t^j}{\sum_{j=0}^{N} q_j t^j}\,, \tag{4.1}$$

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where the coefficients $p_1, p_2, \cdots, p_M, q_0, q_1, \cdots, q_N$, are obtained by solving a system of $M+N+1$ algebraic equations

$$\frac{d^n f^{[M/N]}}{dt^n}(a) = f^{(n)}(a)\,, \qquad n = 0,\,1,\,2,\,\ldots,\,M+N\,. \tag{4.2}$$

The equations (4.2) came from equating the coefficients of $(t-a)^n$ (up to the order $n = M+N$) in the Taylor expansion of the functions $f(t)$ and $f^{[M/N]}(t)$ around some point $t = a$ (which usually is taken at $t = 0$).

Having sketched briefly the method to approximate functions by means of rational functions. Next, we are going to use this method for computing higher level character formulas of pure spinors. Let us start by written the formal expression (3.12) for the partition function of pure spinors like the following

$$Z(q,t) = Z_0(t)\left[1 + \sum_{h=1}^{\infty} f_h(t) q^h\right]\,, \tag{4.3}$$

where the level h function $f_h(t)$ is defined by

$$f_h(t) = \frac{1}{h!}\frac{\partial^h}{\partial q^h}\tilde{Z}(q,t)\big|_{q=0} \quad \text{where} \quad \tilde{Z}(q,t) = \prod_{k=1}^{\infty}\prod_{h=1}^{\infty}(1-q^h t^k)^{-N_k}(1-q^h t^{-k})^{-N_k}. \tag{4.4}$$

As we know by a previous work [7], up to the level $h = 5$, these level h functions are given by rational functions. Therefore, this result is an indication that these level h functions can be computed by means of Padé approximants. In fact this is the case as it is shown in the appendix A, the functions $f_1(t),\ f_2(t),\ f_3(t),\ \cdots$ can be calculated using Padé approximants. As our main result, we have noted that these functions can be written like the following

$$f_h(t) = \frac{\sum_{i=0}^{2h+6} C_{i,h} t^i}{t^{h+2}(1+4t+t^2)}\,, \tag{4.5}$$

where the value of the coefficients $C_{i,h}$ up to the level $h = 12$ are shown in tables 1 and 2.

We have defined the values of the $C_{i,0}$ coefficients such that the level zero function is defined as $f_0(t) = 1$. It is interesting to note that the coefficients $C_{i,h}$ satisfy the following identities

$$C_{i,h} = C_{2h+6-i,h}\,, \tag{4.6}$$

$$\sum_{i=0}^{h'} \phi_{h'-i} C_{i,h} = -\sum_{i=0}^{h} \phi_{h-i} C_{i,h'}\,, \tag{4.7}$$

which can be derived by using the two symmetries of the partition function (3.13), (3.14) and verified by using the coefficients shown in tables 1 and 2. The coefficient ϕ_m is generated by

$$\frac{Z_0(t)}{1+4t+t^2} = \sum_{n=0}^{\infty} \phi_n t^n\,, \tag{4.8}$$

i	$C_{i,0}$	$C_{i,1}$	$C_{i,2}$	$C_{i,3}$	$C_{i,4}$	$C_{i,5}$	$C_{i,6}$	$C_{i,7}$
0	0	0	−1	−16	−126	−672	−2772	−9504
1	0	0	12	146	920	3996	13440	37224
2	1	0	−67	−536	−2411	−7616	−18358	−35184
3	4	46	248	822	1852	3270	7752	33356
4	1	40	319	1200	1745	− 5944	−48147	−179648
5	0	46	628	4114	17000	48206	91948	87730
6	0	0	319	3720	21767	82112	210717	326760
7	0	0	248	4114	32356	162662	585464	1575690
8	0	0	−67	1200	21767	162552	778424	2706944
9	0	0	12	822	17000	162662	977032	4215020
10	0	0	−1	−536	1745	82112	778424	4454624
11	0	0	0	146	1852	48206	585464	4215020
12	0	0	0	−16	−2411	−5944	210717	2706944
13	0	0	0	0	920	3270	91948	1575690
14	0	0	0	0	−126	−7616	−48147	326760
15	0	0	0	0	0	3996	7752	87730
16	0	0	0	0	0	−672	−18358	−179648
17	0	0	0	0	0	0	13440	33356
18	0	0	0	0	0	0	−2772	−35184
19	0	0	0	0	0	0	0	37224
20	0	0	0	0	0	0	0	−9504

Table 1: List of coefficients $C_{i,h}$ up to level $h = 7$.

and it is given explicitly by the formula

$$\phi_n = \frac{(1+n)(2+n)(3+n)(4+n)(5+n)^2(6+n)(7+n)(8+n)(9+n)}{2^7 \cdot 3^4 \cdot 5^2 \cdot 7}\,. \tag{4.9}$$

The importance of the identity (4.7) is as follows. If we know the coefficients $C_{i,0}$, $C_{i,1}$, $\cdots$, $C_{i,h'}$, it is possible to compute explicitly the coefficients $C_{0,h}$, $C_{1,h}$, $\cdots$, $C_{h',h}$. For instance, setting $h' = 0$ in equation (4.7), we get

$$C_{0,h} = -\sum_{i=0}^{h} \phi_{h-i} C_{i,0}\,, \tag{4.10}$$

using (4.9) and the value of the coefficients $C_{i,0}$ given in the table 1 into the equation (4.10), we obtain

$$C_{0,h} = \frac{(1-h)h(1+h)^2(2+h)^2(3+h)^2(4+h)(5+h)}{2^6 \cdot 3^3 \cdot 5^2 \cdot 7}\,. \tag{4.11}$$

By employing the same steps given above, setting $h' = 1$ in equation (4.7), we arrive to the following expression for the $C_{1,h}$ coefficient

$$C_{1,h} = \frac{(h-1)h(1+h)^2(2+h)(3+h)(4+h)(108+10h+12h^2-h^3)}{2^5 \cdot 3^3 \cdot 5 \cdot 7}\,. \tag{4.12}$$

i	$C_{i,8}$	$C_{i,9}$	$C_{i,10}$	$C_{i,11}$	$C_{i,12}$
0	−28314	−75504	−184041	−416416	−884884
1	87912	180180	320892	484770	565136
2	−55368	−77968	−130185	−342472	−1118117
3	145512	513680	1480688	3596898	7511244
4	−467078	−900256	−1189750	−468240	2940853
5	−112192	−651084	−1221496	23356	8349688
6	−14878	−1971392	−7447790	−17913424	−30692216
7	3130008	3975312	77136	−15954844	−51076344
8	7136292	14067968	18009420	783936	−74353372
9	13953544	36499868	75237248	115010006	94216072
10	18453761	59453552	153557340	318143976	504911177
11	21308252	82467920	255938464	651539178	1363603964
12	18453761	88624848	330202624	999457424	2512598839
13	13953544	82467920	366326624	1301105402	3825279040
14	7136292	59453552	330202624	1398947880	4802902081
15	3130008	36499868	255938464	1301105402	5216743428
16	−14878	14067968	153557340	999457424	4802902081
17	−112192	3975312	75237248	651539178	3825279040
18	−467078	−1971392	18009420	318143976	2512598839
19	145512	−651084	77136	115010006	1363603964
20	−55368	−900256	−7447790	783936	504911177
21	87912	513680	−1221496	−15954844	94216072
22	−28314	−77968	−1189750	−17913424	−74353372
23	0	180180	1480688	23356	−51076344
24	0	−75504	−130185	−468240	−30692216
25	0	0	320892	3596898	8349688
26	0	0	−184041	−342472	2940853
27	0	0	0	484770	7511244
28	0	0	0	−416416	−1118117
29	0	0	0	0	565136
30	0	0	0	0	−884884

Table 2: List of coefficients $C_{i,h}$ up to level $h = 12$.

Finally, it would be important to find an explicit expression for a general coefficient $C_{i,h}$ (for all $h \geq 0$ and $i \geq 0$). It is clear that if we know explicitly $C_{i,h}$, it should be possible to write a compact expression for the complete pure spinor partition function

$$Z(q,t) = \frac{1+t}{(1-t)^{11}} \sum_{h=0}^{\infty} \sum_{i=0}^{2h+6} C_{i,h} t^{i-h-2} q^h \,, \tag{4.13}$$

where the factor $(1+t)/(1-t)^{11}$, in front of our formula (4.13), comes from substitution of equations (3.5) and (4.5) into the equation (4.3).

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5. Summary and discussions

We have given in detail a prescription for computing the partition function of pure spinors. This prescription is mainly based on the knowledge of the zero mode part $Z_0(t)$ of the partition function. From this level zero character formula, we have extracted the ghosts multiplicities N_k, and using the moments $(\sum_k k^{s+1} N_k)$ of those multiplicities, by employing a novel application of Padé approximants, we were able to compute higher level character formulas $Z_h(t)$ of pure spinors (up to the twelfth mass-level $h = 12$). We have found that our results are in agreement with the results found in [7] (up to the fifth mass-level $h = 5$) where the fixed point technique was used.

We mention a subtle computational issue related to the fixed point formula [7, 8]. In general for SO($2d$) pure spinors the number of fixed points is 2^{d-1} and the complexity of summing over these fixed points (as it was also noted in [8]) grows exponentially with $N = 2^{d-1}$. On the other hand the computations shown in this letter are less complicated, so our technique can be used as an alternative (to the fixed point technique) easier (computationally) way to get the character formulas for higher-dimensional pure spinors.

So far, in this work, we have computed the partition function without the spin dependence on the states. Spin dependence is crucial if we want to prove that the full partition function (including the contribution of the worldsheet matter sector) correctly reproduces the light cone superstring spectrum [7]. Therefore, it would be interesting to know the character formula with the spin dependence in the ghosts-for-ghosts scheme. We leave this issue as a future work.

It would be nice to see whether our technique can be applied to other constrained systems like strings moving on algebraic surfaces. Another possible application would be the computation of the partition function of eleven-dimensional pure spinors, this can be an interesting issue because an attempt at quantization of the supermembrane was given by using eleven-dimensional pure spinors [19].

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A. Computation of higher level character formulas

Higher level character formulas $Z_h(t)$ can be obtained from the formal expression (3.12) as follows. Performing a Taylor expansion of the expression (3.12) around $q = 0$, we have

$$\begin{aligned} Z(q,t) &= Z_0(t) + \sum_{h=1}^{\infty} \frac{q^h}{h!} \frac{\partial^h}{\partial q^h} Z(q,t)\Big|_{q=0} \\ &= Z_0(t) \left[1 + \sum_{h=1}^{\infty} f_h(t) q^h \right] , \end{aligned} \tag{A.1}$$

where the level h function $f_h(t)$ has been defined as the expression (4.4). To obtain these level h functions, we are going to use a method based on Padé approximants. Let us explain our method by computing in detail the level one function $f_1(t)$.

From the expression (4.4), we derive the following expression for the level one function

$$f_1(t) = \sum_{k=1}^{\infty} N_k (t^k + t^{-k}) \,, \tag{A.2}$$

expanding the r.h.s. of (A.2) around $t = 1$ and keeping terms up to some order (relevant for the computations to be done next), we get

$$f_1(t) \, = 2\sum_{k=1}^{\infty} N_k + (t-1)^2 \sum_{k=1}^{\infty} k^2 N_k - (t-1)^3 \sum_{k=1}^{\infty} k^2 N_k + \cdots \tag{A.3}$$

Applying the formula (3.9) to find the even moments $\sum_k N_k$, $\sum_k k^2 N_k$ and replacing them into the equation (A.3), we obtain

$$f_1(t) = 22 + 4(t-1)^2 - 4(t-1)^3 + \cdots \tag{A.4}$$

Using Padé approximants, we express the function $f_1(t)$ as a rational function

$$f_1(t) \cong f_1^{[M/N]}(t) = \frac{1 + \sum_{j=1}^{M} p_j t^j}{\sum_{j=0}^{N} q_j t^j} \,, \tag{A.5}$$

for instance, as a pedagogical illustration let us compute explicitly the [2/1] Padé approximant of $f_1(t)$

$$f_1^{[2/1]}(t) = \frac{1 + p_1 t + p_2 t^2}{q_0 + q_1 t} \,, \tag{A.6}$$

expanding the r.h.s. of (A.6) around $t = 1$, we get

$$\begin{aligned} f_1^{[2/1]}(t) \, &= \frac{1 + p_1 + p_2}{q_0 + q_1} + \frac{p_1 q_0 + 2p_2 q_0 - q_1 + p_2 q_1}{(q_0 + q_1)^2}(t-1) \\ &+ \frac{p_2 q_0^2 - p_1 q_0 q_1 + q_1^2}{(q_0 + q_1)^3}(t-1)^2 - \frac{p_2 q_1 q_0^2 - p_1 q_0 q_1^2 + q_1^3}{(q_0 + q_1)^4}(t-1)^3 + \cdots \end{aligned} \tag{A.7}$$

Equating the coefficients of $(t-1)^0$, $(t-1)^1$, $(t-1)^2$, $(t-1)^3$ in equations (A.4) and (A.7), we get 4 equations for the unknown coefficients p_1, p_2, q_0, q_1

$$\begin{aligned} \frac{1 + p_1 + p_2}{q_0 + q_1} &= 22 \,, \\ \frac{p_1 q_0 + 2p_2 q_0 - q_1 + p_2 q_1}{(q_0 + q_1)^2} &= 0 \,, \\ \frac{p_2 q_0^2 - p_1 q_0 q_1 + q_1^2}{(q_0 + q_1)^3} &= 4 \,, \\ \frac{p_2 q_1 q_0^2 - p_1 q_0 q_1^2 + q_1^3}{(q_0 + q_1)^4} &= 4 \,, \end{aligned} \tag{A.8}$$

$[M/N]$	$p_1, p_2, \cdots, p_M$	$q_0, q_1, \cdots, q_N$
[2/1]	7/2, 1	0, 1/4
[2/2]	20/23, 1	1/46, 2/23, 1/46
[3/1]	1, 18/25, −2/25	1/50, 1/10
[1/3]	43/23	75/5566, 875/5566, −135/2783, 1/121
[3/2]	20/23, 1, 0	1/46, 2/23, 1/46
[2/3]	20/23, 1	1/46, 2/23, 1/46, 0
[3/3]	20/23, 1, 0	1/46, 2/23, 1/46, 0
[4/4]	20/23, 1, 0, 0	1/46, 2/23, 1/46, 0, 0

Table 3: List of coefficients obtained for higher Padé approximants.

solving these system of equations (A.8) we obtain

$$p_1 = \frac{7}{2}\,, \quad p_2 = 1\,, \quad q_0 = 0\,, \quad q_1 = \frac{1}{4}\,. \tag{A.9}$$

Computations of higher Padé approximants follows in the same way as it was shown above. The results of these computations are shown in table 3.

As we can see by explicit computations, the Padé approximants are approaching to the rational function $(46+40t+46t^2)/(1+4t+t^2)$, and therefore we take this function as being the level one function $f_1(t)$

$$f_1(t) = \frac{46+40t+46t^2}{1+4t+t^2}\,. \tag{A.10}$$

By multiplying this function (A.10) with the level zero character $Z_0(t)$, we get

$$Z_1(t) = \frac{46-144t+116t^2+16t^3-16t^5-116t^6+144t^7-46t^8}{(t-1)^{16}}\,, \tag{A.11}$$

and therefore, we correctly reproduce the level one character formula given in [7, 9].

For the next level $h=2$, by using the same strategy shown above, we have found that the Padé approximant computation gives the following result for the level two function

$$f_2(t) = \frac{-1+12t-67t^2+248t^3+319t^4+628t^5+319t^6+248t^7-67t^8+12t^9-t^{10}}{t^4(1+4t+t^2)}.$$

By multiplying this level two function $f_2(t)$ with the level zero character $Z_0(t)$, we correctly reproduce the level two character formula found in [7].

Computation of higher level functions $f_h(t)$ by means of Padé approximants, suggest us that these functions can be written like

$$f_h(t) = \frac{\sum_{i=0}^{2h+6} C_{i,h} t^i}{t^{h+2}(1+4t+t^2)}\,. \tag{A.12}$$

We have computed the $C_{i,h}$ coefficients up to the level $h=12$, the results are given in the tables 1 and 2 of section 4. Multiplying the functions $f_h(t)$ with the level zero character formula $Z_0(t)$, we obtain the characters $Z_h(t)$. We have compared our first five character formulas with the formulas given in [7] and we have found agreement.

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